

Numerical results of solving 3D inverse scattering problem with non-over-determined data

Cong Van

Department of Mathematics, Kansas State University,
Manhattan, KS 66506, USA
congvan@math.ksu.edu

Abstract

We consider the 3D inverse scattering problem with non-over-determined scattering data. The data are the scattering amplitude $A(\beta, \alpha_0, k)$ for all $\beta \in S_\beta^2$, where S_β^2 is an open subset of the unit sphere S^2 in \mathbb{R}^3 , $\alpha_0 \in S^2$ is fixed, and for all $k \in (a, b)$, $0 \leq a < b$. The basic uniqueness theorem for this problem belongs to Ramm [4]. In this paper, a numerical method is given for solving this problem and the numerical results are presented.

1 Introduction

Let's first consider the direct scattering problem with a potential:

$$(\nabla^2 + k^2 - q(x))u = 0 \quad \text{in } \mathbb{R}^3, \quad (1)$$

$$u = e^{ik\alpha \cdot x} + v, \quad (2)$$

$$v = A(\beta, \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \frac{x}{r} = \beta, \quad (3)$$

where $\alpha, \beta \in S^2$ are the directions of the scattered wave and incident wave correspondingly, S^2 is the unit sphere, $k^2 > 0$ is energy, $k > 0$ is a constant, $A(\beta, \alpha, k)$ is the scattering amplitude or scattering data, which can be measured, and $q(x) \in Q$, where Q is a set of C^1 -smooth real-valued compactly supported functions, $q = 0$ for $\max_j |x_j| \geq R$, $R > 0$ is a real constant.

The direct scattering problem (1) - (3) has a unique solution (see, e.g., [1]).

Consider now the *inverse scattering problem*: find the potential $q(x) \in Q$ from the scattering data $A(\beta, \alpha, k)$.

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The uniqueness of the inverse scattering problem with fixed-energy data is proved by A.G.Ramm [1], i.e., $q(x) \in Q$ is uniquely determined by the scattering data $A(\beta, \alpha, k_0)$ for a fixed $k = k_0 > 0$ and all $\alpha, \beta \in S^2$. A.G.Ramm also gave a method for solving the inverse scattering problem with fixed-energy data and obtained an error estimate for the solution for exact data and also for noisy data, [3].

In this paper, we give a numerical method for solving the inverse scattering problem with non-over-determined data, i.e., find $q(x) \in Q$ from the scattering data $A(\beta, \alpha_0, k)$ for a fixed $\alpha_0 \in S^2$, all $\beta \in S^2$, and all $k \in (a, b)$, $0 \leq a < b$. The basic uniqueness theorem for this problem belongs to Ramm [4]. In Section 2, the idea of the numerical method with its difficulties is presented. In Section 3, the numerical procedure is presented and in Section 4, the numerical results is obtained.

2 Inversion method

Let D be the support of $q(x)$, then the unique solution to (1) - (3) is

$$u(x, k) = e^{ik\alpha_0 \cdot x} - \int_D g(x, y, k) q(y) u(y, k) dy. \quad (4)$$

Let $h(x, k) = q(x)u(x, k)$ then (4) implies

$$h(x, k) = q(x)e^{ik\alpha_0 \cdot x} - q(x) \int_D g(x, y, k) h(y, k) dy. \quad (5)$$

Equations (3) and (4) imply the formula for the scattering amplitude:

$$-4\pi A(\beta, k) = \int_D e^{-ik\beta \cdot y} h(y, k) dy. \quad (6)$$

Equation (5) implies the formula for finding $q(x)$:

$$q(x) = h(x, k) [e^{ik\alpha_0 \cdot x} - \int_D g(x, y, k) h(y, k) dy]^{-1}. \quad (7)$$

The idea of our inversion method is: first discretize (6) to find $h(y, k)$ from the amplitude data $A(\beta, k)$ then use $h(y, k)$ to find $q(x)$ from (7).

Let us partition D into P small cubes with volumn Δ_p , $1 \leq p \leq P$. Let y_p is any point inside the small cube Δ_p . Choose P different $k_m \in (a, b)$, $1 \leq m \leq P$ and choose P different vectors $\beta_j \in S^2$, $1 \leq j \leq P$. Then discretize (6) and get

$$-4\pi A(\beta_j, k_m) = \sum_{p=1}^P e^{-ik_m \beta_j \cdot y_p} h_{pm} \Delta_p, \quad 1 \leq j, m \leq P, \quad (8)$$

where $h_{pm} = h(y_p, k_m)$. Solve the linear system (8) numerically then use equation (7) to find the values of the unknown potential $q(x_p)$

$$q(x_p) = h_{pm} \left[e^{ik_m \alpha_0 \cdot x_p} - \sum_{p'=1, p' \neq p}^P g(x_p, y_{p'}, k_m) h_{p'm} \Delta_{p'} \right]^{-1}, \quad 1 \leq p \leq P. \quad (9)$$

Note that the right hand side of (9) should not depend on m or j . This independence is an important requirement in numerical solution of the inverse scattering problem, a compatibility condition for the data. This requirement is automatically satisfied for the limiting integrals equation (7).

The values of $q(y_p)$ essentially determine the unknown potential $q(x)$ if P is large. This potential is unique by the uniqueness theorem in [4].

Note that one can choose β_j and k_m so that the determinant of the system (8) is not equal to zero, so that the system is uniquely solvable, but the difficulty is that the system (8) is very ill-conditioned because it comes from an integral equation of the first kind with an analytic kernel. We use the dynamical system method (DSM) in [2] to solve the ill-posed system (8).

3 Numerical procedure and results

In practice, one can measure the scattering data (with noises) experimentally. For our numerical experiments, we need to construct the noisy scattering data $A(\beta_j, k_m)$.

3.1 Constructing noisy scattering data

Given a potential $q(x)$, let's first construct the exact scattering data $A^*(\beta_j, k_m)$. We partition D into P small cubes and discretize equation (4) to get

$$u(x_p, k_m) = e^{ik_m \alpha_0 \cdot x_p} - \sum_{j=1}^P g(x_p, y_j, k) q(y_j) u(y_j, k) \Delta_j. \quad (10)$$

One solves this linear system to get $u(x_p, k_m)$, then the exact scattering data can be found by the following formula:

$$A^*(\beta_j, k_m) = -\frac{1}{4\pi} \sum_{p=1}^P e^{-ik\beta_j \cdot y_p} q(y_p) u(y_p, k_m) \Delta_p. \quad (11)$$

Then one can randomly perturb each $A^*(\beta_j, k_m)$, $1 \leq j \leq P$ by $\delta^* = \text{const} > 0$ to get the noisy scattering data $A(\beta_j, k_m) = A^*(\beta_j, k_m) \pm \delta^*$, here the plus or minus sign is chosen alternatively. So, $\|A(\beta_j, k_m) - A^*(\beta_j, k_m)\| = \delta$ for some $\delta > 0$.

3.2 Numerical procedure

The following steps are implemented in each experiment

1. Choose D , α_0 , P , $q(x)$, k_m and δ^* .
2. Use the procedure in Section 3.1 to obtain the noisy scattering data $A(\beta_j, k_m)$ with noise $\delta = \|A(\beta_j, k_m) - A^*(\beta_j, k_m)\|$.
3. Try different values of k_m so that the determinant of the system in (8) is not zero. Let k be the found value of k_m .
4. Solve the linear algebraic system (8) to get $h_p = h(y_p, k)$. Here we use the DSM method in [2] with the noise δ .
5. Construct the potential $q^*(x_p)$ from the equation (9) with $k_m = k$:

$$q^*(x_p) = h_p \left[e^{ik\alpha_0 \cdot x_p} - \sum_{p'=1, p' \neq p}^P g(x_p, y_{p'}, k) h_{p'} \Delta_{p'} \right]^{-1}, \quad 1 \leq p \leq P.$$

6. Find the relative error between the reconstructed potential $q^*(x_p)$ and the original potential $q(x)$:

$$\text{err} = \frac{\|q(x) - q^*(x)\|}{\|q(x)\|} = \sqrt{\frac{\sum_{p=1}^P |q(x_p) - q^*(x_p)|^2 \Delta_p}{\sum_{p=1}^P |q(x_p)|^2 \Delta_p}}. \quad (12)$$

4 Numerical results

In these experiments, we choose D to be the unit cube around the origin, $P = 1000$, $\alpha_0 = (1, 0, 0)$, and $50 \leq k_m \leq 100$.

4.1 For constant potential

Although the inverse scattering problem with constant potential is not interesting, we use the constant potential to test our inversion method.

In this experiment, we take $q(x) = 10$. The following results are obtained:

Constant potential $q(x) = 10$		
δ^*	$\delta = \ A(\beta_j, k_m) - A^*(\beta_j, k_m)\ $	relative error
0.04	0.4348	0.0566
0.02	0.2174	0.0037
0.01	0.1087	0.00065

Table 1: Numerical results for constant potential $q(x) = 10$.

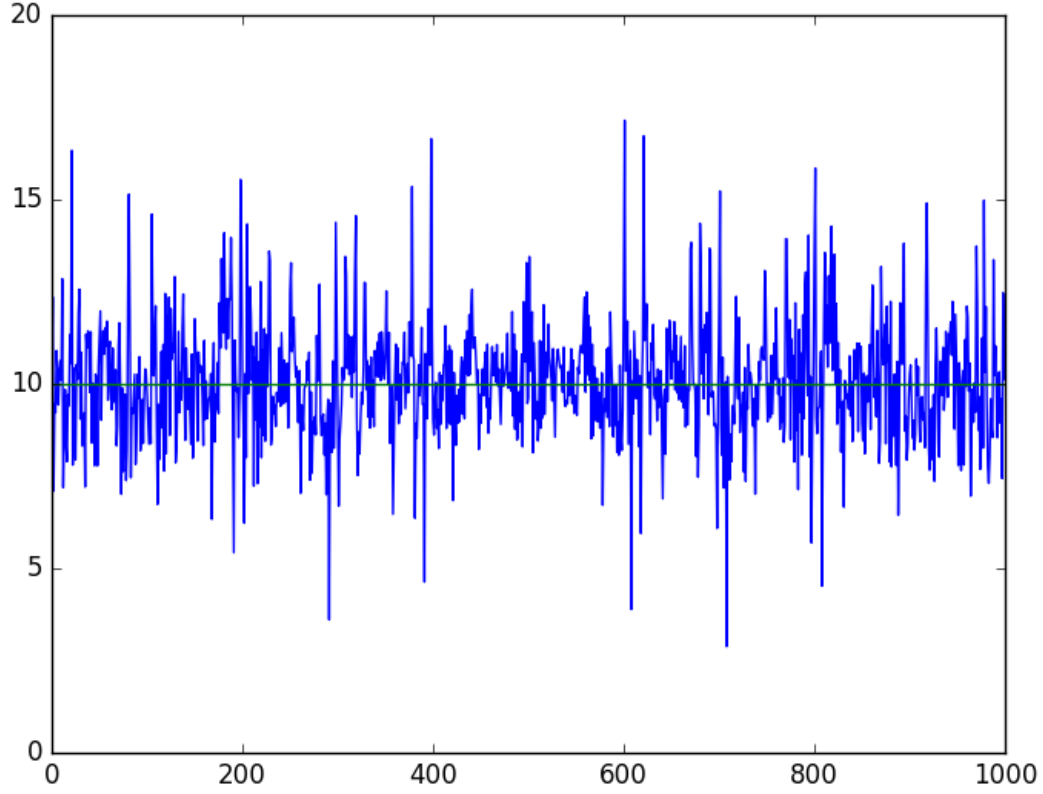


Figure 1: Constructed potential vs original constant potential $q(x) = 10$ when $\delta^* = 0.01$

4.2 For potential $q(x) = \frac{\exp(-|x|)}{|x|}$

In this experiment, we take $q(x) = \frac{\exp(-|x|)}{|x|}$. The following results are obtained:

Potential $q(x) = \frac{\exp(- x)}{ x }$		
δ^*	$\delta = A(\beta_j, k_m) - A^*(\beta_j, k_m) $	relative error
0.04	0.0806	0.1284
0.02	0.0403	0.0547
0.01	0.0201	0.0367

Table 2: Numerical results for the potential $q(x) = \frac{\exp(-|x|)}{|x|}$.

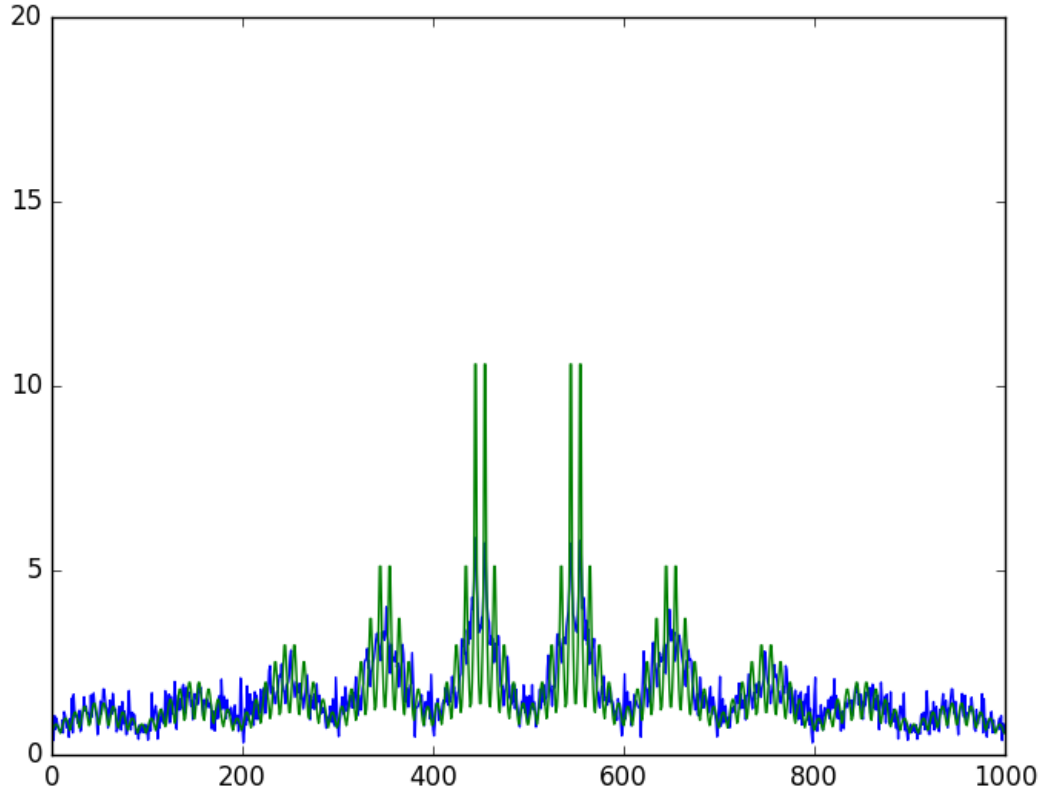


Figure 2: Constructed potential vs original potential $q(x) = \frac{\exp(-|x|)}{|x|}$ when $\delta^* = 0.01$

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